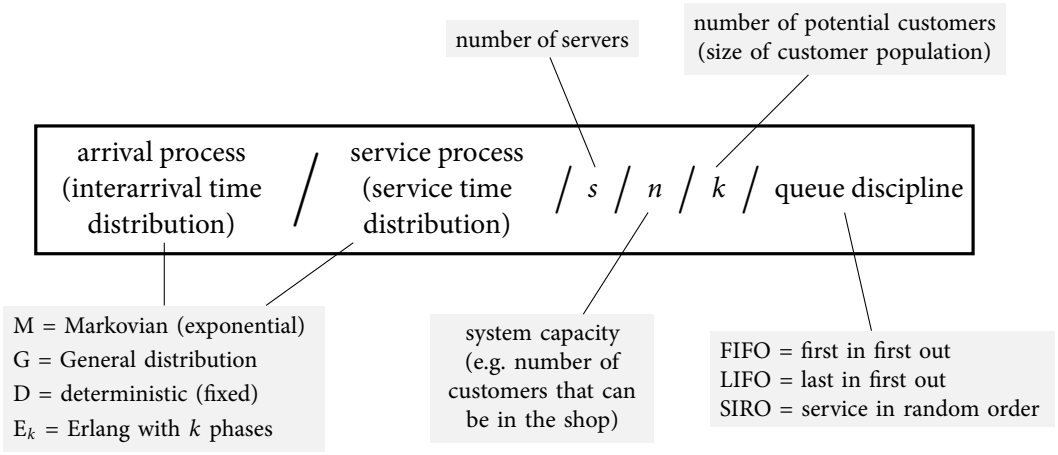
Lesson 16. Standard Queueing Models

1 Standard queueing notation



- If not specified, default values:

$$n = \infty \qquad k = \infty \qquad \text{queueing discipline} = \text{FIFO}$$

Example 1. What does M/M/∞ mean? Model an M/M/∞ queue as a birth-death process by specifying the arrival and service rates.

Example 2. What does M/M/s mean? Model an M/M/s queue as a birth-death process by specifying the arrival and service rates.

2 The M/M/∞ queue

- Let's apply the formulas for the steady-state probabilities:

$$\pi_j = \frac{d_j}{D} \quad \text{for } j = 0, 1, 2, \ldots \quad \text{where} \quad d_0 = 1, \quad d_j = \frac{\lambda_0 \lambda_1 \cdots \lambda_{j-1}}{\mu_1 \mu_2 \ldots \mu_j} \quad \text{for } j = 1, 2, \ldots, \quad D = \sum_{i=0}^{\infty} d_i$$

- We can simplify d_j:

- Therefore, we can rewrite D:

- As a result, the steady-state probabilities for a M/M/∞ queue are:

- The number of customers in steady state L is

Example 3. Recall the Massive Mall case: we want to determine the number of parking spaces needed for the new mall by pretending that parking is unlimited, and then investigating how many spaces are sufficient to satisfy demand a large fraction of the time. Assume customers arrive according to a Poisson process with an arrival rate of 1000 per hour. In addition, suppose the time that a customer spends at the mall is exponentially distributed with a mean of 3 hours.

a. What is the expected number of cars in the parking lot?
b. What is the expected time a car spends in the parking lot?
c. What is the minimum number of parking spaces needed to hold all cars 99.9% of the time?

3 The M/M/s queue

- Steady-state probabilities:

$$\rho = \frac{\lambda}{s\mu} \qquad \pi_0 = \left[\left(\sum_{j=0}^{s} \frac{(s\rho)^j}{j!} \right) + \frac{s^s \rho^{s+1}}{s!(1-\rho)} \right]^{-1} \qquad \pi_j = \begin{cases} \dfrac{(\lambda/\mu)^j}{j!}\pi_0 & \text{for } j = 1, 2, \ldots, s \\ \dfrac{(\lambda/\mu)^j}{s!s^{j-s}}\pi_0 & \text{for } j = s+1, s+2, \ldots \end{cases}$$

- Expected number of customers in queue and expected delay:

$$\ell_q = \frac{\pi_s \rho}{(1-\rho)^2} \qquad w_q = \frac{\ell_q}{\lambda}$$

- Expected number of customers in the system and expected waiting time:

$$\ell = \ell_q + \frac{\lambda}{\mu} \qquad w = \frac{\ell}{\lambda}$$

Example 4. The Darker Image copy shop is considering adding a second photocopier. Suppose customers arrive according to a Poisson process with rate 4 customers per hour, and that the service time of each photocopier is exponentially distributed with a mean of 12 minutes. Compare the expected delay of customers when there is 1 copier vs. when there are 2 copiers.

4 The G/G/s queue

- When interarrival times and service times are not Markovian (exponentially distributed), things get much harder
- Usually, we resort to **simulation** to understand these queues – this is the focus of SA421!
- However, we can still use results from Markovian queues to approximate performance measures
- **Whitt's approximation (1983)**
  - Consider a G/G/s queue with

$$\begin{aligned} \text{interarrival times } G \quad &\text{with} \quad E[G] = 1/\lambda \quad \Leftrightarrow \quad \lambda = 1/E[G] \\ \text{service times } X \quad &\text{with} \quad E[X] = 1/\mu \quad \Leftrightarrow \quad \mu = 1/E[X] \end{aligned}$$

  - **Squared coefficients of variation:**

$$\varepsilon_G = \frac{\text{Var}[G]}{E[G]^2} \qquad \varepsilon_X = \frac{\text{Var}[X]}{E[X]^2}$$

  - Let

    $\hat{w}_q$ = expected delay in this G/G/s queue
    w_q = expected delay in a M/M/s queue with the same arrival rate λ and same service rate μ

  - We can approximate $\hat{w}_q$ using w_q and the squared coefficients of variation:

$$\hat{w}_q \approx \frac{\varepsilon_G + \varepsilon_X}{2} w_q$$

  - We can use this with Little's law (both versions) to find approximations of ℓ_q, ℓ, and w
  - This approximation works well when ε_G, ε_X, and $\rho = \frac{\lambda}{s\mu}$ are "close" to 1
- Note: This is one of many approximations that have been proposed for G/G/s queues
- For more details, see SMAS pp. 262-263

Example 5. Consider the Darker Image case from Example 4 again. Suppose now that the service time of each photocopier is uniformly distributed between 3 and 21 minutes. Now compare the expected delay of customers when there is 1 copier vs. when there are 2 copiers.

5 Exercises

Problem 1. The Kalman Theater Group (KTG) is building a movie theater mega-complex. They have decided that there will automatic ticket kiosks in front of a single first-come-first-served queue, but they still need to decide how many kiosks to include in the complex design. Based on data that they have collected from their other theaters in similar markets, they have estimated that customers arrive at the kiosks at a rate of 5 per minute, and customers can be served in 2 minutes on average. KTG's standard configuration for mega-complexes in similar markets is 12 kiosks. You have been asked to evaluate this standard configuration.

Assume that the interarrival times and the service times are exponentially distributed.

a. What standard queueing model fits this setting best?
b. What is the traffic intensity in this queueing system?
c. What is the long-run expected fraction of time that all kiosks are unoccupied?
d. Over the long-run, what is the expected time a customer waits in line?

Problem 2 (SMAS Exercise 8.8)**.** Four Guys Burger and Fries currently operates a single drive-thru window. Customers place their order at an intercom at the back of the parking lot. After placing their order, customers pull up to the drive-thru and wait in line. Since filling the orders takes muich longer than placing an order, queueing is caused primarily by the process of filling orders. You have been hired as analyst to determine whether to have one or two employees to fill the orders. Note that orders will still be filled one at a time; having two employees simply increases the rate at which the orders can be filled. With one employee filling orders, the average service time is 2 minutes per customer. With a second employee, the average service time is reduced to 1.25 minutes per customer. Customers arrive at a rate of 24 per hour.

a. Determine the average delay and waiting time for the one- and two-employee systems. Carefully state your modeling assumptions.
b. Determine the percentage of time that the workers are completely idle for each system.
c. Suppose that when there are more than three cars waiting for the drive-thru window (including the one receiving their order), then any additional arrivals cannot reach the intercom. In other words, there is room for only three cars between the window and the intercom. Given this information, evaluate the one- and two-employee systems in terms of their effect on the intercom.

Problem 3 (SMAS Exercise 8.10)**.** The Hessian Hotel offers a customer support phone line. Presently, there are two employees who handle calls individually, and the average call takes 3 minutes to complete. When both support people are busy, customers who call are placed in a hold queue in which they listen to music and information about the Hessian Hotel. To date, the company has not observed any reneging, but they are worried that the introduction of a music festival nearby may significantly increase the arrival rate of calls, causing the delays to become excessive. Presently calls arrive at a rate of 20 per hour during peak hours.

a. Develop a queueing model of the hotel's customer support phone line.
b. How large can the arrival rate become before the company will be forced to add another support person just to keep up with calls?
c. With just two employees, how large can the arrival rate become before the expected time a customer spends on hold exceeds 4 minutes?
d. With just two employees, how large can the arrival rate become before the percentage of time that there are more than five people in the hold queue exceeds 15% of the time?
e. Develop a model that includes the fact that customers will only wait, on average, 5 minutes before reneging from the hold queue.